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LETTER TO THE EDITOR

An exact solution in non-linear oscillations

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Abstract. The equations of motion for a particle moving in one dimension under the influence of forces which vary cubically with displacement, have been solved exactly, in terms of elliptic functions. The result is not original, but the simple derivation I shall use is not readily available in the literature.

In about 1860, Weierstrass obtained a solution to the differential equation

$$\ddot{x}+\alpha+\beta x+\gamma x^2+\varepsilon x^3=0 \tag{1}$$

where α, β, γ and ε are constants. He gave the proof during a lecture, but it was left to one of the students, Biermann, to publish it, as part of an inaugural dissertation given at the University of Berlin, in 1865 [1]. Some years later, it was given as an example in the textbooks of Whittaker and Watson [2] and Greenhill [3] with the reader required to supply his own proof. I shall use notation and ideas from Greenhill's book in order to establish the formula.

We begin by rewriting the differential equation (1) into its integral representation:

$$t=\int_{x(0)}^{x(t)}\frac{\mathrm{d}z}{\sqrt{az^4+4bz^3+6cz^2+4dz+e}} \tag{2}$$

where

$$\begin{aligned}&a=-\tfrac{1}{2}\varepsilon \qquad b=-\tfrac{1}{6}\gamma \qquad c=-\tfrac{1}{6}\beta \qquad d=-\tfrac{1}{2}\alpha\\ &e=\dot{x}^2(0)+2\alpha x(0)+\beta x^2(0)+\tfrac{2}{3}\gamma x^3(0)+\tfrac{1}{2}\varepsilon x^4(0).\end{aligned} \tag{3}$$

This is a special case of the more general problem in which the lower bound of the integral is also a function of time, say $y(t)$. It is easier to proceed from the general to the specific, so we shall begin at the equation

$$t=\int_{y(t)}^{x(t)}\frac{\mathrm{d}z}{\sqrt{Z}} \tag{4}$$

using the notation that X, Y and Z are quartics with the same coefficients, but in variables x, y and z respectively.

Taking any fixed value of t, it may be shown that equation (4) implies an algebraic relation between the allowed values of $x(t)$ and $y(t)$. To do this we imagine that t, $x(t)$ and $y(t)$ each depend on a fourth, dummy, variable Ω. So we may differentiate (4) to obtain

$$\frac{\mathrm{d}t}{\mathrm{d}\Omega}=\frac{1}{\sqrt{X}}\frac{\mathrm{d}x}{\mathrm{d}\Omega}-\frac{1}{\sqrt{Y}}\frac{\mathrm{d}y}{\mathrm{d}\Omega} \tag{5}$$

where all square roots are taken to be positive. If t is a constant, equation (5) implies the differential relation

$$\frac{\mathrm{d}x}{\sqrt{X}}-\frac{\mathrm{d}y}{\sqrt{Y}}=0 \tag{6}$$

which can be integrated, by a simple method due to Lagrange, to give

$$s=\frac{1}{4}\left(\frac{\sqrt{X}+\sqrt{Y}}{x-y}\right)^2-\frac{1}{4}a(x+y)^2-b(x+y)-c. \tag{7}$$

In (7) s is a constant of integration, which has been defined in such a way as to give it a special meaning in the context of this problem. To investigate its meaning, we differentiate once more with respect to the dummy variable Ω:

$$\frac{\mathrm{d}s}{\mathrm{d}\Omega}=\frac{\partial s}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\Omega}+\frac{\partial s}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\Omega}. \tag{8}$$

It may be shown that this can be written as

$$\frac{\mathrm{d}s}{\mathrm{d}\Omega}=-\frac{A(x,y)}{\sqrt{X}}\frac{\mathrm{d}x}{\mathrm{d}\Omega}+\frac{A(x,y)}{\sqrt{Y}}\frac{\mathrm{d}y}{\mathrm{d}\Omega} \tag{9}$$

where:

$$\begin{aligned}A(x,y)=\{[(ay^3+3by^2+3cy+d)x+(by^3+3cy^2+3dy+e)]\sqrt{X}\\ +[(ax^3+3bx^2+3cx+d)y+(bx^3+3cx^2+3dx+e)]\sqrt{Y}\}/(x-y)^3.\end{aligned} \tag{10}$$

This too has its special meaning, for if we rewrite

$$S=4s^3-(ae-4bd+3c^2)s-(ace+2bcd-ad^2-eb^2-c^3) \tag{11}$$

in terms of x and y, we may verify that

$$S=A^2(x,y) \tag{12}$$

and so we may rewrite equation (9) as

$$\frac{1}{\sqrt{X}}\frac{\mathrm{d}x}{\mathrm{d}\Omega}-\frac{1}{\sqrt{Y}}\frac{\mathrm{d}y}{\mathrm{d}\Omega}=-\frac{1}{\sqrt{S}}\frac{\mathrm{d}s}{\mathrm{d}\Omega}. \tag{13}$$

Integrating with respect to Ω, and combining the result with equation (4), we have proved the following important relation:

$$t=\int_{y(t)}^{x(t)}\frac{\mathrm{d}z}{\sqrt{Z}}=\int_{s(t)}^{\infty}\frac{\mathrm{d}r}{\sqrt{R}} \tag{14}$$

where R is the cubic in equation (11), but with variable r. Now the meaning of $s(t)$ becomes clear, for the right-hand integral is in Weierstrassian normal form, and so it is equivalent to the relation

$$s(t)=\wp(t;\, g_2, g_3) \tag{15}$$

where we have used the customary notation

$$g_2=ae-4bd+3c^2 \tag{16}$$

and

$$g_3=ace+2bcd-ad^2-eb^2-c^3. \tag{17}$$

These are well known in the theory of algebraic equations as projective invariants of the roots of the quartic Z.

Furthermore, differentiating (14) with respect to $s(t)$, we obtain the relation

$$S(t)=\mathscr{P}'^2(t;\, g_2, g_3). \tag{18}$$

Now, in order to completely solve the problem, all that remains is to rearrange (7) to give $x(t)$ as a function of $s(t)$ and $y(t)$. To begin we note that (7) may be written in the form

$$s(t)=\frac{F(x,y)+\sqrt{X}\sqrt{Y}}{2(x-y)^2} \tag{19}$$

where, using Taylor's theorem,

$$F(x,y)=Y+\tfrac{1}{2}Y'(x-y)+\tfrac{1}{12}Y''(x-y)^2. \tag{20}$$

Rearranging (19), squaring, and using the expansion

$$XY=Y^2+YY'(x-y)+\tfrac{1}{2}YY''(x-y)^2+\tfrac{1}{6}YY'''(x-y)^3+\tfrac{1}{24}YY''''(x-y)^4 \tag{21}$$

we obtain the following expression:

$$\left(4s^2-\frac{sY''}{3}+\frac{(Y'')^2}{144}-\frac{YY''''}{24}\right)(x-y)^4+\left(-2sY'+\frac{Y'Y''}{12}-\frac{YY'''}{6}\right)(x-y)^3$$
$$+\left(-4sY+\frac{(Y')^2}{4}-\frac{YY''}{3}\right)(x-y)^2=0. \tag{22}$$

Reducing this to a quadratic, and solving by the usual method, we obtain:

$$(x-y)=\left[\frac{Y'}{2}\left(s-\frac{Y''}{24}\right)+\frac{YY'''}{24}+\sqrt{Y}\sqrt{S}\right]\left[2\left(s-\frac{Y''}{24}\right)^2-\frac{YY''''}{48}\right]^{-1} \tag{23}$$

where we have taken positive square roots, to concur with (19), and used the identities

$$g_2=\frac{YY''''}{24}-\frac{Y'Y'''}{24}+\frac{(Y'')^2}{48} \tag{24}$$

and

$$g_3=\frac{Y'Y''Y'''}{576}-\frac{Y(Y''')^2}{576}-\frac{(Y'')^3}{1728}+\frac{YY''Y''''}{288}-\frac{(Y')^2Y''''}{384}. \tag{25}$$

Using (15) and (18), setting $y(t)=x(0)$, and writing

$$Y[x(0)]=F[x(0)] \tag{26}$$

we are left with the general solution of equation (1):

$$x(t)=x(0)+\left[F^{1/2}[x(0)]\mathscr{P}'(t;\, g_2, g_3)+\frac{F'[x(0)]}{2}\left(\mathscr{P}(t;\, g_2, g_3)-\frac{F''[x(0)]}{24}\right)\right.$$
$$\left.+\frac{F[x(0)]F'''[x(0)]}{24}\right]$$
$$\times\left[2\left(\mathscr{P}(t;\, g_2 g_3)-\frac{F''[x(0)]}{24}\right)^2-\frac{F[x(0)]F''''[x(0)]}{48}\right]^{-1}. \tag{27}$$

In fact this expression may be greatly simplified if we assume the boundary condition

$$\dot{x}(0)=0 \tag{28}$$

because in this circumstance, equations (2) and (3) show that $x(0)$ is automatically a root of the quartic. So (27) may be written

$$x(t)=x(0)+\frac{F'[x(0)]}{4(\wp(t;\,g_2,g_3)-\frac{1}{24}F''[x(0)])}. \tag{29}$$

References

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